SUCCESSIVE DIFFERENTIATION

❖ **Introduction to topic :**

It is extension of differentiation of one variable function.
Weightage for university exam: 08 Marks
No. of lectures required to teach: 04 hrs

❖ **Definition of Successive differentiation**:

➢ Consider,

A one variable function,
y = f(x) (**x** is independent variable and **y** depends on **x**.)
Here if we make any change in **x** there will be a related change in **y**.
This change is called derivative of **y** w.r.t. **x**. denoted by $f'(x)$ **or** y_1 **or** y' **or** $\frac{dy}{dx}$ called first order derivative of **y** w.r.t. **x.**

$$f''(x) = (f'(x))' = \frac{d^2y}{dx^2} = y'' = y_2$$ is called second order derivative of **y** w.r.t **x.**

It gives rate of change in y_1 w.r.t. rate of change in **x.**

➢ Similarly,

Third derivative of **y** is denoted by y_3 **or** $f'''(x)$ **or** $\frac{d^3y}{dx^3}$ or y''' and
So on……………………..
(Above derivatives exist because, If **y = f(x),** then $y_1 = g(x)$, where **g(x)** is some function of **x** depends on **f(x)**
for e.g. if **y = sinx** then $y_1 = \cos x$,hence $y_2 = -\sin x$ and so on………..)
Thus,
Derivatives of **f(x) (or f)** w.r.t. x are denoted by, $f'(x), f''(x), f'''(x), \ldots\ldots, f^{(n)}(x), \ldots\ldots$
Above process is called successive differentiation of **f (x)** w.r.t. **x** and $f', f'', f''', \ldots\ldots, f^{(n)}$ are called successive derivatives of **f.**

➢ **$f^{(n)}(x)$ denotes n^{th} derivative of f.**

➢ **Notations:**

Successive derivatives of **y** w.r.t. **x** are also denoted by,

1. $y_1, y_2, y_3, \ldots\ldots y_n, \ldots\ldots\ldots\ldots$ **or**
2. $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots \frac{d^ny}{dx^n}, \ldots\ldots\ldots$ **or**
3. $f'(x), f''(x), f'''(x), \ldots\ldots, f^{(n)}(x), \ldots\ldots$ **or**
4. $y', y'', y''', \ldots\ldots y^{(n)}, \ldots\ldots$ **or**
5. $Dy, D^2y, D^3y, \ldots\ldots, D^ny, \ldots\ldots$

Where **D** denotes $\frac{d}{dx}$.

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➢ Value of nth derivative of **y = f(x) at x =a** is denoted by ,

$\mathbf{f^{n}(a), y_n(a), or} \left(\frac{d^n y}{dx^n}\right)_{x=a}$

(i.e. value can be obtained by just replacing **x** with **a** in $\mathbf{f^{n}(x)}$.)

❖ **List of formulas (01) :**

Sr no.	Function	n^{th} derivative
01	$y = e^{ax}$	$y_n = a^n e^{ax}$
02	$y = b^{ax}$	$y_n = a^n b^{ax} (\log_e b)^n$
03	$y = (ax + b)^m$	(i) if m is integer greater than n or less than (-1) then, $y_n = m(m-1)(m-2)\ldots\ldots(m-n+1)\, a^n (ax + b)^{m-n}$
		(ii) if m is less then n then, $y_n = 0$
		(iii) if m = n then, $y_n = a^n\, n!$
		(iv) if m = -1 then , $y_n = \frac{(-1)^n\, n!\, a^n}{(ax + b)^{n+1}}$
		(v) if m = -2 then , $y_n = \frac{(-1)^n (n+1)!\, a^n}{(ax + b)^{n+1}}$
04	y = log (ax +b)	$y_n = \frac{(-1)^{n-1} (n-1)!\, a^n}{(ax + b)^{n}}$

❖ **Problems Based On Above Formulas :**

1. Obtain 5^{th} derivative of e^{2x}.
2. Obtain 3^{rd} derivative of 3^{5x}.
3. Obtain 4^{th} derivative of $(2x +3)^5$
4. Obtain 4^{th} derivative of $(2x +3)^3$
5. Obtain 4^{th} derivative of $(2x +3)^4$
6. Obtain 4^{th} derivative of $\frac{1}{(2x +3)}$

❖ **n^{th} derivatives of reciprocal of polynomials** (n^{th} **derivatives of functions which contain polynomials in denominators)** :

Consider

$$\mathbf{y = \frac{ax+b}{cx^2+dx+e} \quad or \quad y = \frac{1}{cx^2+dx+e}}$$

To find nth derivative of above kind function first obtain partial fractions of **f(x)** or **y**.

To get partial fractions:

If $\mathbf{y = \frac{1}{cx^2+dx+e}}$ then first factorize $\mathbf{cx^2+dx+e}$.

Let **(fx +g). (hx +i**) be factors then $\mathbf{y = \frac{1}{(fx+g).(hx+i)}}$

Find **A** & **B** such that $\mathbf{y = \frac{A}{fx+g} + \frac{B}{hx+i}}$

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obtain n^{th} derivatives of above fractions separately and add them, answer will give n^{th} derivative of **y**.

❖ **Note:**

If polynomial in denominator is of higher Degree then we will have more factors .(Do the same process for all the factors).

➢ If $\mathbf{y = \dfrac{1}{(fx+g)^2.(hx+i)}}$ then use factors $\mathbf{y = \dfrac{A}{(fx+g)^2} + \dfrac{B}{hx+i} + \dfrac{C}{fx+g}}$

❖ Problems Based On Above Formulas & notes :

Obtain n^{th} derivatives of followings:

(1) $\dfrac{1}{a^2 - x^2}$ (2) $\dfrac{ax+b}{cx+d}$ (3) $\dfrac{x}{(x-1)(x-2)(x-3)}$ (4) $\dfrac{x^2}{(x+2)(2x+3)}$ (5) $\dfrac{8x}{(x+2)(x-2)^2}$

(6) $x \log \dfrac{(x-1)}{(x+1)}$ (7) $\dfrac{1}{x^2+a^2}$ (8) $\dfrac{1}{x^2+x+1}$.

❖ ASSIGNMENT (01) :

Obtain n^{th} derivatives of followings:

(1) $\dfrac{a-x}{a+x}$ (2) $\dfrac{1}{(x-1)^2(x-2)}$ (3) $\dfrac{x^4}{(x-1)(x-2)}$ (4) $\dfrac{x}{a^2+x^2}$

❖ List of formulas (02) :

Sr no.	Function	n^{th} derivative
01	$y = \sin(ax+b)$	(i)$y_n = a^n \sin(ax + b + n\,\pi/_2)$ (ii)if b =0 , a =1 then $y = \sin x$ & $y_n = \sin(x + n\,\pi/_2)$
02	$y = \cos(ax+b)$	(i)$y_n = a^n \cos(ax + b + n\,\pi/_2)$ (ii)if b =0, a =1 then $y = \cos x$ & $y_n = \cos(x + n\,\pi/_2)$
03	$y = e^{ax} \sin(bx + c)$	(i)$y_n = r^n\, e^{ax} \sin(bx + c + n\Theta)$ where $r = (a^2+b^2)^{1/2}$ $\Theta = \tan^{-1}(b/a)$
04	$y = e^{ax} \cos(bx + c)$	(i)$y_n = r^n\, e^{ax} \cos(bx + c + n\Theta)$ where $r = (a^2+b^2)^{1/2}$ $\Theta = \tan^{-1}(b/a)$

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❖ **Problems Based On Above Formulas :**

1. Obtain 4^{th} derivative of sin(3x+5).
2. Obtain 3^{rd} derivative of $e^{2x}\cos 3x$

❖ **Problems Based On Above Formulas :**

Obtain n^{th} derivatives of followings:

(1) $\sin x \sin 2x$ (2) $\sin^2 x\cos^3 x$ (3) $\cos^4 x$ (4) $e^{2x}\cos x \sin^2 2x$

❖ **ASSIGNMENT (02) :**

Obtain n^{th} derivatives of followings:

(1) $\cos x\cos 2x\cos 3x$ (2) $\sin^4 x$ (3) $e^{-x}\cos^2 x \sin x$

❖ **Some Problems (Problems Of Special Type) Based On Above All (01& 02) formulas:**

(1) For $y = \dfrac{x^3}{x^2-1}$,

Show that, $$\left(\frac{d^n y}{dx^n}\right)_{x=0} = \begin{cases} 0 & \text{if n is even} \\ (-n) & \text{if n is odd integer greater than 1} \end{cases}$$

(2) If y = cosh2x, show that

$y_n = 2^n \sinh 2x$, when n is odd.
$= 2^n \cosh 2x$, when n is even.

(3) Find n^{th} derivative of following:

(i) $\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$ (ii) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ (iii) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ (iv) $\tan^{-1}x$

(4) If u = sinnx + cosnx, show that

$$u_r = n^r\left[1+(-1)^r \sin 2nx\right]^{\frac{1}{2}}$$

where u_r denotes the r^{th} derivative of u with respect to x.

(5) If $\mathbf{I_n} = \dfrac{d^n}{dx^n}(x^n \log x)$,

Prove that $\mathbf{I_n} = n\,I_{n-1} + (n-1)!$,
Hence show that

$$\mathbf{I_n} = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \dots\dots + \frac{1}{n}\right)$$

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- **Leibnitz's theorem(only statement):**

If **y = u .v**,

where **u** & **v** are functions of x possessing derivatives of nth order then,

$\mathbf{y_n = nC_0u_nv + nC_1u_{n-1}v_1 + nC_2u_{n-2}v_2 + \ldots + nC_ru_{n-r}v_r + \ldots + nC_nuv_n}$

where, $\mathbf{nCr = \dfrac{n!}{r!(n-r)!}}$

Properties:

1) **nCr = nCn-r**
2) $\mathbf{nC_0 = 1 = nCn}$
3) $\mathbf{nC_1 = n = nCn\text{-}1}$

- **Note:**

Generally we can take any function as u and any as v.(If y = u .v)
But take v as the function whose derivative becomes zero after some order.

- **Problems Based On Leibnitz's theorem:**

Obtain n^{th} derivatives of followings:

(1) $x^3 \log x$ (2) $\dfrac{x^n}{x+1}$ (3) $x^2 e^x \cos x$

- **ASSIGNMENT (03) :**

Obtain n^{th} derivatives of followings (using Leibnitz's theorem):

(1) $x^2 \log x$ (2) $x^2 e^x$ (3) $x \tan^{-1}x$.

- **Solved Problems (Problems Of Special Type) Based On Leibnitz's theorem:**

(1) If $y = \sin(m\sin^{-1}x)$
Then prove, $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n = 0$

(2) If $y = \cot^{-1}x$,
Then prove, $(1+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n = 0$

(3) If $y^{1/m} + y^{-1/m} = 2x$
Then prove, $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n = 0$

- **ASSIGNMENT (04) :**

(1) If $\cos^{-1}\left(\dfrac{y}{b}\right) = \log\left(\dfrac{x}{n}\right)^n$ then prove, $x^2y_{n+2+}(2n+1)xy_{n+1}+2n^2y_n = 0$

(2) If $y = (x^2-1)^n$ then prove, $(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n = 0$

(3) If $y = \tan^{-1}\left(\dfrac{a+x}{a-x}\right)$ then prove, $(a^2+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n$.

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