SUCCESSIVE DIFFERENTIATION

✿ Introduction to topic :
   It is extension of differentiation of one variable function.
   Weightage for university exam: 08 Marks
   No. of lectures required to teach: 04 hrs

✿ Definition of Successive differentiation:

➢ Consider,
   A one variable function,
   \( y = f(x) \) (x is independent variable and y depends on x.)
   Here if we make any change in x there will be a related change in y.
   This change is called derivative of y w.r.t. x. denoted by \( f'(x) \) or \( y_1 \) or \( y' \) or \( \frac{dy}{dx} \) called first order derivative of y w.r.t. x.

\[ f''(x) = (f'(x))' = \frac{d^2y}{dx^2} = y_2 \] is called second order derivative of y w.r.t x.

It gives rate of change in \( y_1 \) w.r.t. rate of change in x.

➢ Similarly,
Third derivative of y is denoted by \( y_3 \) or \( f'''(x) \) or \( \frac{d^3y}{dx^3} \) or \( y''' \) and so on…………..

(Above derivatives exist because, If \( y = f(x) \), then \( y_1 = g(x) \), where \( g(x) \) is some function of x depends on f(x)
for e.g. if \( y = \sin x \) then \( y_1 = \cos x \), hence \( y_2 = -\sin x \) and so on………….)

Thus,
Derivatives of \( f(x) \) (or f) w.r.t. x are denoted by, \( f'(x), f''(x), f'''(x), \) ……….. \( f^{(n)}(x), \) ………..

Above process is called successive differentiation of \( f(x) \) w.r.t. x and \( f', f'', f''', \) ……….\( f^{(n)} \) are called successive derivatives of f.

➢ \( f^{(n)}(x) \) denotes \( n^{th} \) derivative of f.

➢ Notations:

   Successive derivatives of y w.r.t. x are also denoted by,
   1. \( y_1, y_2, y_3, \ldots \ldots \ldots \ldots \ldots y_n, \ldots \ldots \ldots \ldots \ldots \) or

   2. \( \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots \ldots \ldots \ldots \ldots \frac{d^n y}{dx^n} \) or

   3. \( f'(x), f''(x), f'''(x), \ldots \ldots \ldots \ldots \ldots f^{(n)}(x) \) or

   4. \( y', y'', y''', \ldots \ldots \ldots \ldots \ldots y^{(n)} \) or

   5. \( D^y, \ D^2y, \ D^3y, \ldots \ldots \ldots \ldots \ldots \ D^n y \) or

   Where \( D \) denotes \( \frac{d}{dx} \).
Value of nth derivative of \( y = f(x) \) at \( x = a \) is denoted by,
\[
f^n(a) \quad y_n(a), \quad \text{or} \quad \left( \frac{d^n y}{dx^n} \right)_{x=a}
\]
(i.e. value can be obtained by just replacing \( x \) with \( a \) in \( f^n(x) \).)

**List of formulas (01):**

<table>
<thead>
<tr>
<th>Sr no.</th>
<th>Function</th>
<th>( n^{th} ) derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>( y = e^{ax} )</td>
<td>( y_n = a^n e^{ax} )</td>
</tr>
<tr>
<td>02</td>
<td>( y = b^{ax} )</td>
<td>( y_n = a^n b^{ax} (\log_b b)^n )</td>
</tr>
</tbody>
</table>
| 03     | \( y = (ax + b)^m \) | (i) if \( m \) is integer greater than \( n \) or less than \(-1\) then,
\[
y_n = m(m-1)(m-2)\ldots (m-n+1) a^n (ax + b)^{m-n}
\]
(ii) if \( m \) is less then \( n \) then, \( y_n = 0 \)
(iii) if \( m = n \) then, \( y_n = a^n n! \)
(iv) if \( m = -1 \) then , \( y_n = (-1)^n n! a^n \)
\[
(ax + b)^{n+1}
\]
(v) if \( m = -2 \) then , \( y_n = (-1)^n (n+1)! a^n \)
\[
(ax + b)^{n+1}
\]
| 04     | \( y = \log (ax +b) \) | \( y_n = (-1)^{n-1} (n-1)! a^n \)
\[
(ax + b)^{n}
\]

**Problems Based On Above Formulas:**

1. Obtain 5\(^{th} \) derivative of \( e^{2x} \).
2. Obtain 3\(^{rd} \) derivative of \( 3^x \).
3. Obtain 4\(^{th} \) derivative of \( (2x +3)^5 \).
4. Obtain 4\(^{th} \) derivative of \( (2x +3)^3 \).
5. Obtain 4\(^{th} \) derivative of \( (2x +3)^4 \).
6. Obtain 4\(^{th} \) derivative of \( \frac{1}{(2x +3)} \).

**\( n^{th} \) derivatives of reciprocal of polynomials (\( n^{th} \) derivatives of functions which contain polynomials in denominators):**

Consider
\[
y = \frac{ax +b}{cx^2+dx+e} \quad \text{or} \quad y = \frac{1}{cx^2+dx+e}
\]
To find \( n^{th} \) derivative of above kind function first obtain partial fractions of \( f(x) \) or \( y \).
To get partial fractions:

If \( y = \frac{1}{cx^2+dx+e} \) then first factorize \( cx^2+dx+e \).

Let \((fx +g) (hx+i)\) be factors then
\[
y = \frac{1}{(fx +g)(hx+i)}
\]
Find \( A \) & \( B \) such that
\[
y = \frac{A}{fx +g} + \frac{B}{hx +i}
\]
obtain \(n\)th derivatives of above fractions separately and add them, answer will give \(n\)th derivative of \(y\).

\[\text{Note:}\]
If polynomial in denominator is of higher Degree then we will have more factors .(Do the same process for all the factors).

\[\text{If } y = \frac{1}{(fx +g)^2.(hx +i)} \text{ then use factors } y = \frac{A}{(fx+g)^2} + \frac{B}{hx + i} + \frac{C}{fx + g}\]

\[\text{Problems Based On Above Formulas & notes :}\]

Obtain \(n\)th derivatives of followings:

1. \(\frac{1}{a^2 - x^2}\)
2. \(\frac{ax + b}{cx + d}\)
3. \(\frac{x}{(x-1)(x-2)(x-3)}\)
4. \(\frac{x^2}{(x+2)(2x+3)}\)
5. \(8x\)
6. \(x \log \left(\frac{x-1}{x+1}\right)\)
7. \(\frac{1}{x^2+a^2}\)
8. \(\frac{1}{x^2+x+1}\)

\[\text{ASSIGNMENT (01) :}\]

Obtain \(n\)th derivatives of followings:

1. \(\frac{a-x}{a+x}\)
2. \(\frac{1}{(x-1)(x-2)}\)
3. \(\frac{x^4}{(x-1)(x-2)}\)
4. \(\frac{x}{a^2 + x^2}\)

\[\text{List of formulas (02) :}\]

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<tr>
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| 01     | \(y = \sin(ax + b)\) | (i)\(y_n = a^n \sin(ax + b + n\pi/2)\)  
(ii)if \(b \neq 0\), \(a \neq 1\) then \(y = \sin x \& y_n = \sin(x + n\pi/2)\) |
| 02     | \(y = \cos(ax + b)\) | (i)\(y_n = a^n \cos(ax +b + n\pi/2)\)  
(ii)if \(b = 0\), \(a = 1\) then \(y = \cos x \& y_n = \cos(x + n\pi/2)\) |
| 03     | \(y = e^{ax} \sin(bx + c)\) | (i)\(y_n = r^n e^{ax} \sin(bx + c + n\Theta)\)  
where \(r = (a^2 + b^2)^{1/2}\)  
\(\Theta = \tan^{-1}(b/a)\) |
| 04     | \(y = e^{ax} \cos(bx + c)\) | (i)\(y_n = r^n e^{ax} \cos(bx + c + n\Theta)\)  
where \(r = (a^2 + b^2)^{1/2}\)  
\(\Theta = \tan^{-1}(b/a)\) |
Problems Based On Above Formulas:

1. Obtain 4th derivative of \(\sin(3x+5)\).
2. Obtain 3rd derivative of \(e^{2x}\cos3x\).

Problems Based On Above Formulas:

Obtain \(n^{th}\) derivatives of followings:

1. \(\sin x \sin2x\)
2. \(\sin^2 x \cos3x\)
3. \(\cos^4 x\)
4. \(e^{2x} \cos x \sin^2 2x\)

ASSIGNMENT (02):

Obtain \(n^{th}\) derivatives of followings:

1. \(\cos x \cos2x \cos3x\)
2. \(\sin^4 x\)
3. \(e^{\cos^2 x} \sin x\)

Some Problems (Problems Of Special Type) Based On Above All (01 & 02) formulas:

(1) For \(y = \frac{x^3}{x^2-1}\),
Show that, \(\left(\frac{d^n y}{dx^n}\right)_{x=0} = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-n) & \text{if } n \text{ is odd integer greater than 1} \end{cases}\)

(2) If \(y = \cosh2x\), show that
\(y_n = 2^n \sinh2x\), when \(n\) is odd.
\(-2^n \cosh2x\), when \(n\) is even.

(3) Find \(n^{th}\) derivative of following:

(i) \(\tan^{-1}\left(\frac{1-x}{1+x}\right)\)
(ii) \(\sin^{-1}\left(\frac{2x}{1+x^2}\right)\)
(iii) \(\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\)
(iv) \(\tan^{-1}x\)

(4) If \(u = \sin nx + \cos nx\), show that
\(u_r = n\left[1 + (-1)^r \sin 2nx\right]^{\frac{1}{2}}\)
where \(u_r\) denotes the \(r^{th}\) derivative of \(u\) with respect to \(x\).

(5) If \(I_n = \frac{d^n}{dx^n} (x^n \log x)\),
Prove that \(I_n = n I_{n-1} + (n-1)!\),
Hence show that
\(I_n = n! \left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)\)
Leibnitz’s theorem (only statement):

If \( y = u \cdot v \), where \( u \) & \( v \) are functions of \( x \) possessing derivatives of \( n \)th order then,
\[
y_n = nC_0u_nv + nC_1u_{n-1}v_1 + nC_2u_{n-2}v_2 + \ldots + nC_{n-r}u_{n-r}v_r + \ldots + nC_nv
\]

where, \( nCr = \frac{n!}{r!(n-r)!} \)

Properties:
1) \( nCr = nCn-r \)
2) \( nC_0 = 1 = nCn \)
3) \( nC_1 = n = nCn-1 \)

Note:
Generally we can take any function as \( u \) and any as \( v \). (If \( y = u \cdot v \))
But take \( v \) as the function whose derivative becomes zero after some order.

Problems Based On Leibnitz’s theorem:

Obtain \( n \)th derivatives of followings:

1) \( x^3 \log x \) 
2) \( \frac{x^n}{x+1} \) 
3) \( x^2 \cdot e^x \cdot \cos x \)

ASSIGNMENT (03):

Obtain \( n \)th derivatives of followings (using Leibnitz’s theorem):

1) \( x^2 \log x \) 
2) \( x^2 \cdot e^x \) 
3) \( x \cdot \tan^{-1} x \)

Solved Problems (Problems Of Special Type) Based On Leibnitz’s theorem:

1) If \( y = \sin (m \sin^{-1} x) \)
Then prove, \((1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n = 0 \)

2) If \( y = \cot^{-1} x \),
Then prove, \((1+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n = 0 \)

3) If \( y^{1/m} + y^{-1/m} = 2x \)
Then prove, \((x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n = 0 \)

ASSIGNMENT (04):

1) If \( \cos^{-1} \left( \frac{y}{b} \right) = \log \left( \frac{3}{x} \right)^n \) then prove, \( x^2y_{n+2}-(2n+1)xy_{n+1}+2n^2y_n = 0 \)

2) If \( y = (x^2-1)^n \) then prove, \((x^2-1)y_{n+2}+2xy_{n+1}+n(n+1)y_n = 0 \)

3) If \( y = \tan^{-1} \left( \frac{a+x}{a-x} \right) \) then prove, \((a^2+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n = 0 \).