PARTIAL DIFFERENTIATION
AND
APPLICATIONS OF PARTIAL DIFFERENTIATION

❖ **Introduction to topic** :

Here, we are going to discuss continuity & derivatives of functions having more than one independent variables.

Weightage for university exam: 16Marks

No. of lectures required to teach: 12hrs

❖ **Definition of function having more than one independent variables**:

If three variables x, y, z are related in such a way that z depends upon the values of x&y, then z is called function of two variables x&y.
It is denoted z = f(x, y).

Similarly, w = g(x, y, z) is a function of three variables x, y, z.

U = $f(x_1, x_2, x_3, ..., x_n)$ is a function of n variables $x_1, x_2, x_3, ..., x_n$.

❖ **Definition of continuity of two variable functions:**

A function f(x, y) is said to be continous at (a ,b) if

(i) f is defined at (a, b)

(ii) $\lim_{(x, y)\to(a, b)} f(x, y)$ exists and

(iii) $\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$ in whatever manner x&y approach a & b respectively.

❖ **Definition of partial derivatives of first order**:

Let z = f(x , y) be a function of two variables .
We can make change in z by three ways.

First: keep y constant and make change in value of x

Second : keep x constant and make change in value of y

Third : make change in value of x&y both simultaneously.

If we make change in value of x , keeping y constant then the relative change in value of z is called partial derivative of z with respect to x. and it is denoted by $\frac{\partial f}{\partial x}$ or f_x or $\frac{\partial z}{\partial x}$. Similarly, partial derivative of z with respect to y is denoted by $\frac{\partial f}{\partial y}$ or f_y or $\frac{\partial z}{\partial y}$. Also, $\frac{\partial f}{\partial x} = f_x = \frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$.

Prepared by Mr.Zalak Patel
Lecturer, Mathematics

$$\frac{\partial f}{\partial y} = fy = \frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y).}{\delta y}$$

❖ **PARTIAL DIFFERENTIATION OF HIGHER ORDER**:

Let z =f(x, y) then $\frac{\partial z}{\partial x} and. \frac{\partial z}{\partial y}$ are first oprder derivatives of z with respect to x &y.

Second order partial derivatives of z with respect to x & y are denoted by

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}, \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{yx}, \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{xy}.$$

Third order partial derivatives of z with respect to x & y are denoted by

$$\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = f_{xxx}, \frac{\partial^3 z}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right) = f_{yyy},$$

$$\frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x \partial y}\right) = f_{yxx}, \frac{\partial^3 z}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y \partial x}\right) = f_{xyy}$$

$$\frac{\partial^3 z}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = f_{yyx}, \frac{\partial^3 z}{\partial y \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x^2}\right) = f_{xxy}.$$

➢ **SOME OTHER NOTATIONS TO DENOTE P.D.** :

$let\ x = f(r,\theta) \& y = g(r,\theta)\ (i.e. x \& y\ both\ are\ the\ functions\ of\ r \& \theta)$

$then\ indirectly\ x\ is\ a\ function\ of\ y\ and\ viseversa.$

$In\ this\ case,$

$\left(\frac{\partial x}{\partial r}\right)_\theta$ $denotes\ derivative\ of\ x\ w.r.t.\ r\ keeping\ \theta\ constant$

$\left(\frac{\partial y}{\partial r}\right)_\theta$ $denotes\ derivative\ of\ y\ w.r.t.\ r\ keeping\ \theta\ constant$

$\left(\frac{\partial x}{\partial \theta}\right)_r$ $denotes\ derivative\ of\ x\ w.r.t.\ \theta\ keeping\ r\ constant$

$\left(\frac{\partial y}{\partial \theta}\right)_r$ $denotes\ derivative\ of\ x\ w.r.t.\ \theta\ keeping\ r\ constant$

$if\ we\ eliminate\ \theta\ from\ above\ function\ we\ will\ get\ x\ as\ a\ function\ of\ y \& r$

$or\ y\ as\ a\ function\ of\ x \& r.$

$\left(\frac{\partial x}{\partial y}\right)_r$ $denotes\ derivative\ of\ x\ w.r.t.\ y\ keeping\ r\ constant$

$\left(\frac{\partial x}{\partial r}\right)_y$ $denotes\ derivative\ of\ x\ w.r.t.\ \theta\ keeping\ r\ constant$

And similarly other derivatives can be defined.

❖ **DERIVATIVES OF FUNCTION OF FUNCTION** :

If z = f(t) and t = g(x,y) then z = fog(x,y) is a composite function of f & g
If we consider z as a function of t alone then we have ordinary derivative of z w.r.t. t which is $\frac{dz}{dt}$, but if we consider z as a composite function of f &g then z will be function of two variables .in this case we can find partial derivatives of z w.r.t. x&y which are $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ .And relation between above derivatives is

$$\frac{\partial z}{\partial x} = \frac{dz}{dt}\cdot\frac{\partial t}{\partial x} \ \& \ \frac{\partial z}{\partial y} = \frac{dz}{dt}\cdot\frac{\partial t}{\partial y}.$$

➢ **CHAIN RULE FOR PARTIAL DIFFERENTIATION** :

(A) If u =f(x ,y) and if x = g(t) &y = h(t) then u is indirectly function of t alone

Hence, $$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad \text{………..(1)}$$

$\frac{du}{dt}$ is called total differential coefficient

(i) If we eliminate dt from both sides we get Total Derivative of u

That can be given by $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad \text{…………..(2)}$$

(ii) If we replace t by x in above (1) then we get

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \quad \text{………..…...(3)}$$

In this case u is a function of x &y and y is function of x. so ultimately u becomes function of x alone.

If u = f (x_1, x_2, x_3, x_4,….,x_n) and all x_1,x_2,x_3,x_4,….x_n are functions of a single variable t.

Then,

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial u}{\partial x_3}\frac{dx_3}{dt} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt} \quad \text{....(4)}$$

And $$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \frac{\partial u}{\partial x_3}dx_3 + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{\partial u}{\partial x_n}dx_n \quad \text{…..(5)}$$

(B) If z =f(x, y) and if x = g(u ,v) &y = h(u ,v) then z is indirectly function of two variables u & v.

Hence,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \quad \text{…………(6)}$$

❖ **Solved Problems Based On Above Discussion** :

1). $If\ u = \tan^{-1}\left(\frac{y}{x}\right)\ then\ verify : \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$

2). $If\ u = x^2 y + y^2 z + z^2 x,\ prove\ that\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2$

3) If $v = (1 - 2xy + y^2)^{\frac{-1}{2}}$,

prove that, (i) $x\dfrac{\partial v}{\partial x} - y\dfrac{\partial v}{\partial y} = y^2 v^3$

$$(ii)\ \frac{\partial}{\partial x}\left[\left(1 - x^2\right)\frac{\partial v}{\partial x}\right] + \frac{\partial u}{\partial y}\left[y^2 \frac{\partial v}{\partial y}\right] = 0$$

4). If $z = \log(x^2 + y^2) + \tan^{-1}\left(\dfrac{y}{x}\right)$,

prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

5). If $z(x + y) = x^2 + y^2$,

prove that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left[1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right].$

6). If $f(x, y) = \dfrac{1}{\sqrt{y}} e^{\frac{-(x-a)^2}{4y}}$,

prove that (i) $\dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial f}{\partial y} = 0$ (ii) $\dfrac{\partial^2 f}{\partial x \partial y} - \dfrac{\partial^2 f}{\partial y \partial x} = 0.$

7). If $V = r^m$, where $r^2 = x^2 + y^2 + z^2$,

provethat $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = m(m+1)r^{m-2}$

8). provethatata point $x = y = z$ of thesurface $x^x y^y z^z = c$, $\dfrac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$

9). If u = e^{xyz} show that $\dfrac{\partial^3 u}{\partial y \partial x \partial z} = -(1 + 3xyz + x^2 y^2 z^2).u$

10). If $\dfrac{x^2}{a^2 + u} + \dfrac{y^2}{b^2 + u} + \dfrac{z^2}{c^2 + u} = 1$

where u is afunction of x, y, z.

prove that, $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial u}{\partial z}\right)^2 = 2\left[x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}\right]$

11). If $u = \dfrac{x + y + z}{\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$

12) If $ux+vy=0$ and $\frac{u}{x}+\frac{v}{y}=1$, then prove that $\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{y^2+x^2}{y^2-x^2}$.

13) If $u=ax+by$ and $v=bx-ay$, then prove that $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial v}{\partial y}\right)_u \left(\frac{\partial y}{\partial v}\right)_x = 1$.

14) If $x^2 = au+bv$ and $y^2 = au-bv$, then prove that $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$.

15) If $x = u\tan v$ and $y = u\sec v$, then prove that $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x$.

16) If $x = r\cos\theta$ and $y = r\sin\theta$, then prove that (i) $\frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r = r\left(\frac{\partial \theta}{\partial x}\right)_y$

17) If $xyz = a^3$, then prove that $\left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial x}{\partial z}\right)_y = -1$.

18) $Find\ \frac{du}{dt}\ when\ \ u=\sin\left(\frac{x}{y}\right), x=e^t\ , y=t^2$

19) $If\ u=f(r,s,t)\ and\ r=\frac{x}{y},\ s=\frac{y}{z},\ t=\frac{z}{x},\ prova\ that\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$

20) $If\ z\ be\ a\ function\ of\ x\ and\ y\ and\ x=e^u+e^{-v}, y=e^{-u}-e^v$

$prove\ that\ \frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$

21) $If\ z=e^{ax+by}\ f(ax-by), prove\ that\ b\frac{\partial z}{\partial x}+a\frac{\partial z}{\partial y}=2baz.$

22) $If\ u=f\left(\frac{x}{z},\frac{y}{z}\right), prove\ that\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$

23) $If\ z=f(x,y), x=u\cosh v\ and\ y=u\sinh v, prove\ that$

$$\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial u}\right)^2-\frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2.$$

24) $find\ the\ total\ differential\ of\ f(x,y,z)=xe^{y^2-z^2}$

25) $If\ z\ is\ a\ function\ of\ x\ and\ y\ and\ u\ \&\ v\ be\ two\ variables\ such\ that$

$u=ix+my, v=ly-mx,\ then\ prove\ that$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(l^2+m^2\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$$

26) $If\ z\ is\ a\ function\ of\ x\ and\ y\ and\ u\ \&\ v\ be\ two\ variables\ such\ that$

$x=e^u\sin v,\ y=e^u\cos v,\ then\ prove\ that$

$$\left(x^2+y^2\right)\ \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$$

27) $If\ x=\sqrt{vw}, y=\sqrt{wu}, z=\sqrt{uv}\ \ and\ \phi\ is\ a\ function\ of\ x,y,z\ then$

$show\ that\ x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}+z\frac{\partial\phi}{\partial z}=u\frac{\partial\phi}{\partial u}+v\frac{\partial\phi}{\partial v}+w\frac{\partial\phi}{\partial w}.$

❖ **IMPLICIT FUNCTION** :

If we can not express x as a function of y or y as a function x even though we have relationship f (x, y) = 0 ,then the function f(x, y) = 0 is called an implicit function.

for e.g. $x^3+y^3+3axy=0$.

➢ **(A) DIFFERENTIATION OF IMPLICIT FUNCTION** :

To find derivative of implicit function we can use following method.
If f(x, y) = 0 is a given implicit function

then assume,

(i) u = f(x, y) and y = g(x)

Hence, by chain rule $\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}$ but f(x, y) =0 = u

$$\therefore 0=f_x+f_y\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=-\frac{f_x}{f_y}$$

(ii) u = f(x, y) and x = h(x)

Hence, by chain rule $\frac{\partial u}{\partial y}=\frac{\partial f}{\partial x}\frac{dx}{dy}+\frac{\partial f}{\partial y}$ but f(x, y) =0 =u

$$\therefore 0=f_x\frac{dx}{dy}+f_y$$

$$\therefore \frac{dx}{dy}=-\frac{f_y}{f_x}$$

- **(B) PARTIAL DIFFERENTIATION OF IMPLICIT FUNCTION** :

f (x, y, z) = 0 represents an implicit function .

To find partial derivative of implicit function we can use following method.
If f(x, y, z) = 0 is a given implicit function
then assume,

(i) u = f(x, y, z) and z = g(x, y)

Hence, by chain rule

$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$ And $\frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}$

But f(x, y, z) = 0 = u

$\therefore 0=f_x+f_z\frac{\partial z}{dx}$. And $\therefore 0=f_y+f_z\frac{\partial z}{dy}$

$\therefore \frac{\partial z}{dx}=-\frac{f_x}{f_z}$ $\therefore \frac{\partial z}{dy}=-\frac{f_y}{f_z}$.

(ii) u = f(x, y, z) and y = h(x, z)

Hence, by chain rule

$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial x}$ And $\frac{\partial u}{\partial z}=\frac{\partial f}{\partial z}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial z}$

But f(x, y, z) =0 = u

$$\therefore 0 = f_x + f_y \frac{\partial y}{dx} \qquad \text{And} \qquad \therefore 0 = f_z + f_y \frac{\partial y}{dz}$$

$$\therefore \frac{\partial y}{dx} = -\frac{f_x}{f_y}. \qquad\qquad \therefore \frac{\partial y}{dz} = -\frac{f_z}{f_y}.$$

(iii) u = f(x, y, z) and x = t (y, z)

Hence, by chain rule

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} \qquad \text{And} \qquad \frac{\partial u}{\partial z} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial z}$$

But f(x, y, z) =0 = u

$$\therefore 0 = f_y + f_x \frac{\partial x}{dy} \qquad\qquad \therefore 0 = f_z + f_x \frac{\partial x}{dz}$$

$$\therefore \frac{\partial x}{dy} = -\frac{f_y}{f_x}. \qquad\qquad \therefore \frac{\partial x}{dz} = -\frac{f_z}{f_x}$$

.

❖ Solved Problems Based On Above Discussion

1) $Find\ \frac{dy}{dx}\ by implicit\ differentiation.$
   (i) $x^3 + y^3 + 3xy = 0.$
   (ii) $(\cos x)^y = (\sin y)^x$
2) $If\ f(x,y,z)=0,\ prove that \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial x}{\partial y}\right)_z = -1$
3) $If\ f(x,y)=0\ and\ \phi(x,y)=0,\ show that \frac{\partial f}{\partial y}.\frac{\partial \phi}{\partial z}.\frac{dz}{dx} = \frac{\partial f}{\partial x}.\frac{\partial \phi}{\partial y}.$
4) $If\ u(x,y)=c_1\ and\ v(x,y)=c_2\ show\ that\ the\ curves\ u = constants$ $and\ v = constants\ intersect\ orthogonally\ if\ \frac{\partial u}{\partial x}.\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.\frac{\partial v}{\partial y} = 0.$

❖ HOMOGENEOUS FUNCTIONS:

A function f of two variables x & y is said to be homogeneous function of degree n if,

$$f(tx,ty)=t^n f(x,y) \ \underline{or} \ f(x,y) = x^n g\left(\frac{y}{x}\right) \ \underline{or} \ f(x,y) = y^n h\left(\frac{x}{y}\right).$$

For e.g. $u=\frac{x^2y^2}{x+y}$. is a homogeneous function of degree 3.

$$\text{Since} \quad u = \frac{x^4\left(\frac{y}{x}\right)^2}{x\left(1+\left(\frac{y}{x}\right)\right)} = x^3 g\left(\frac{y}{x}\right) \quad \underline{\text{or}} \ \ u(tx,ty)= t^3 \frac{x^2y^2}{x+y}.$$

➢ **EULER'S THEOREM** :

Statement :

If u is a homogeneous function of x & y of dergree n then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$$

Proof:

Since u = f(x , y) is a homogeneous function of degree n, it can be Written as

$$u(x,y) = x^n g\left(\frac{y}{x}\right) \ \ldots\ldots\ldots\ldots(i)$$

differentiating (i) *partially w.r.t* x

$$\frac{\partial u}{\partial x} = nx^{n-1} g\left(\frac{y}{x}\right) + x^n g'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$= nx^{n-1} g\left(\frac{y}{x}\right) - x^{n-2} y\, g'\left(\frac{y}{x}\right)$$

hence,

$$x\frac{\partial u}{\partial x} = nx^n g\left(\frac{y}{x}\right) - x^{n-1} y\, g'\left(\frac{y}{x}\right)\ldots\ldots\ldots(ii)$$

differentiating (i) *partially w.r.t* y

$$\frac{\partial u}{\partial y} = x^n g'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= x^{n-1} g'\left(\frac{y}{x}\right)$$

hence,

$$y\frac{\partial u}{\partial y} = x^{n-1} y\, g'\left(\frac{y}{x}\right)\ldots\ldots\ldots(iii)$$

Now, adding (ii) & (iii) we get,

Prepared by Mr.Zalak Patel
Lecturer, Mathematics

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n g\left(\frac{y}{x}\right)$$
$$=nu.$$

➢ **Cor-1**: If u is a homogeneous function of x, y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

➢ **Cor-2**: If z is a homogeneous function of x, y of degree n and z = f(u) ,

Then, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,\frac{f(u)}{f'(u)}.$

➢ **Cor-3**: If z is a homogeneous function of x, y of degree n and then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1].$$

$$where\ \ g(u)=n\frac{f(u)}{f'(u)}.$$

❖ **Solved Problems Based On Above Discussion**

1) verify Euler's theorem for (i) $u = e^{\frac{x}{y}}$

(ii) $u = \left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}}$.

2) If $u = f\left(\frac{x}{y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

3) If $u = \sec^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

4) If $z = xy\, f\left(\frac{y}{x}\right)$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$.

5) If $f(x, y) = \sqrt{x^2 - y^2}\, \sin^{-1}\left(\frac{y}{x}\right)$,

prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x, y)$.

6) If $u = \dfrac{x^2 y^2}{x + y}$

prove that $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} = 2\dfrac{\partial u}{\partial x}$.

7) show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

when (i) $u = \dfrac{x - y}{x + y}$ (ii) $u = \dfrac{xy}{x - y}$ (iii) $u = (x - y)\, f\left(\dfrac{y}{x}\right)$

8) If $u = \cos\sqrt{\dfrac{x - y}{x + y}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

9) If $u = \sin^{-1}\sqrt{\dfrac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{5}} + y^{\frac{1}{5}}}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{20}\tan u$.

10) If $u = \tan^{-1}\sqrt{\dfrac{x^3 + y^3}{x - y}}$, show that (i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$.

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2\sin u \cos 3u$.

11) $If\ \ u = \cos ec^{-1}\sqrt{\dfrac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}}$, $show that$

$$x^2\ \frac{\partial^2 u}{\partial x^2} + 2xy\ \frac{\partial^2 u}{\partial x \partial y} + y^2\ \frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u\left(\frac{13}{12} + \frac{1}{12}\tan^2 u\right)$$

12) $If\ u = f\left(\dfrac{y}{x}\right) + x\phi\left(\dfrac{x}{y}\right) + 4xy, show that\ \ x^2\ \dfrac{\partial^2 u}{\partial x^2} + 2xy\ \dfrac{\partial^2 u}{\partial x \partial y} + y^2\ \dfrac{\partial^2 u}{\partial y^2} = 8xy.$

❖ **Jacobians:**

(1) If u = f(x, y) and v = g(x, y) be differentiable functions then the Jacobian of u & v with respect to x & y is defined and denoted by

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ is given by } J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial x} \end{vmatrix}.$$

(2) If u = f(x, y, z) ,v = g(x, y, z) & w = h(x, y, z) be differentiable functions then the Jacobian of u, v& w with respect to x, y & z is defined and denoted by

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \text{ is given by } J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}.$$

❖ **Chain rule for Jacobians**:

(1) If u = f(x, y) & v = g (x, y) and x = h (r, s) & y = t (r, s) be differentiable functions then

$$\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} . \frac{\partial(x,y)}{\partial(r,s)}.$$

(2) $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} . \dfrac{\partial(x,y,z)}{\partial(r,s,t)} = \dfrac{\partial(u,v.w)}{\partial(r,s,t)}.$

❖ **Note:**

(1) If $J = \dfrac{\partial(u,v)}{\partial(x,y)}$ $and\ J' = \dfrac{\partial(x,y)}{\partial(u,v)}$ then J.J' = 1.

(2) If $J = \dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ $and\ J' = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ then J.J' = 1.

❖ **Solved Problems Based On Above Discussion** :

(1) If x = r cosθ, y = r sinθ ,then show that $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$

(2) If x = $e^v \sec u,\ y = e^v \tan u$, then show that J.J' =1.

(3) If x + y = $2e^\theta \cos\phi,\ x - y = 2i\,e^\theta \sin\phi$, then prove that J.J' =1.

(4) If u = $\frac{x+y}{1-xy}$ and v = $\tan^{-1} x + \tan^{-1} y$, show that $\frac{\partial(u,y)}{\partial(x,y)} = 0$, if $x.y \neq 1$ *also show that* $u = \tan v$.

(5) If $u_1 = f_1(x_1),\ u_2 = f_2(x_1, x_2)\ \&\ u_3 = f_3(x_1, x_2, x_3)$.
Show that $\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = \frac{\partial u_1}{\partial x_1}\cdot\frac{\partial u_2}{\partial x_2}\cdot\frac{\partial u_3}{\partial x_3}$

❖ POWER SERIES EXPANSIONS :

➢ Maclaurin's Expansions for functions of two variables:

Maclaurin series Expansions for the function f(x, y) can be given by

$$f(x,y) = f(0,0) + [\,x\,f_x(0,0) + y\,f_y(0,0)] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\,f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \ldots\ldots \quad \textbf{.... (i)}$$

➢ Taylor's Expansions for functions of two variables:

Taylor series Expansions for the function f(x, y) can be given by

$$f(x,y) = f(a,b) + [(x-a)\,f_x(a,b) + (y-b)\,f_y(a,b)] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \ldots\ldots \quad \textbf{...(ii)}$$

➢ Cor-1:

Putting x = a + h & y = b+ k in above equation (ii) we get,

$$f(a+h, b+k) = f(a,b) + [\,h\,f_x(a,b) + k\,f_y(a,b)] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk\,f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \ldots\ldots \quad \textbf{...(iii)}$$

➢ Cor-2:

Putting a= x & b= y in above equation (iii) we get,

$$f(x+h, y+k) = f(x, y) + [\,x\, f_x(x\,, y) + y\, f_y(x, y)]$$
$$+\frac{1}{2!}\left[x^2\, f_{xx}(x, y) + 2xy\, f_{xy}(x, y) + y^2\, f_{yy}(x, y)\right] \qquad \text{........(iv)}$$
$$+\text{.........................}$$

❖ **Solved Problems Based On Above Discussion** :

(1) Expand e^{xy} in powers of (x-1) & (y-1) .

(2) Expand $f(x, y) = \sin xy$ in powers of (x-1) & $\left(y-\frac{\pi}{2}\right)$ up to second degree terms.

(3) Expand $e^x \log(1+y)$ in powers of x & y up to third degree terms.

(4) Expand $e^{ax} \sin by$ in powers of x & y.

(5) Expand $\tan^{-1}\left(\frac{y}{x}\right)$ in powers of (x-1) & (y-1).

❖ **MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES :**

Working rules to find maximum and minimum values of a function f(x , y):

(1) Find $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$.

(2) Solve simultaneous equations $\frac{\partial f}{\partial x} = 0$ *and* $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 0$ *and* $\frac{\partial f}{\partial y} = 0$ is necessary condition for function f(x ,y) to be maximum or minimum.

(3) let (a_1, b_1) ,(a_2, b_2),…….be solutions of simultaneous equations $\frac{\partial f}{\partial x} = 0$ *and* $\frac{\partial f}{\partial y} = 0$. These points are called optimal points.

(4) Evaluate $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$ at above points.

(5) Use following table to decide that function is either maxima or minima at optimal points.

Sr no.	$rt - s^2$	r	conclusion
1	> 0	< 0	Point of maxima
2	> 0	> 0	Point of minima
3	> 0	$= 0$	Further investigation required
4	$= 0$	any	Further investigation required
5	< 0	any	Point of neither Maxima nor Minima Saddle point

(6) points :

Sr no.	Points	Definition
1	Optimal point	A point where function satisfies $\frac{\partial f}{\partial x} = 0$ *and* $\frac{\partial f}{\partial y} = 0$
2	Extreme point	A point where function takes its Maximum or Minimum value.
3	Point of maxima	A point where function takes its Maximum value.
4	Point of minima	A point where function takes its Minimum value.
5	Saddle point	A point where function is neither Maxima nor Minima

(7) values :

Sr no.	Points	Definition
1	Optimal value	Value of function at optimal point
2	Maximum value	Value of function at Point of maxima
3	Minimum value	Value of function at Point of minima

❖ **Problems Based On Above Discussion :**

Find Maximum and Minimum values for the following.

(1) $x^2 + y^2 + 6x + 12$

(2) $x^2 + 2xy + 2y^2 + 2x + y$

(3) $x^3 + y^3 - 3axy$

(4) $2(x^2 - y^2) - x^4 + y^4$

(5) $\sin x + \sin y + \sin(x+y) \qquad x \geq 0, y \leq \frac{\pi}{2}.$

(6) $x^2 y + xy^2 - axy$

(7) $x^3 + y^3 - 63(x+y) + 12xy$

(8) $\sin x \sin y \sin(x+y) \qquad x > 0, y < \pi.$

❖ **LAGRANGE'S METHOD OF UNDETERMIND MULTIPLIERS :**

This method is useful to find extreme values of a function

$$u = f(x, y, z)$$

subject to the condition $\phi(x, y, z) = 0$

Working rules to find maximum and minimum values of a function

$$u = f(x, y, z)$$

Subject to the condition $\phi(x, y, z) = 0$

(1) Form the Lagrangian equation
$u = f(x, y, z) + \lambda \phi (x, y, z)$.

(2) Solve equations $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$ & $\frac{\partial u}{\partial z} = 0$ along with ϕ (x, y, z) = 0 and find the values of x, y, z & λ.
The values of x, y, z so obtained will give the extreme values of f (x, y, z).

❖ **Problems Based On Above Discussion :**

(1) Find the maximum value of xyz subject to the restriction x +y +z = a.
(2) Divide 30 into three parts such that the continued product of the first, the square of the second and the cube third be a maximum.
(3) Divide 120 into three parts so that the sum of the products taken two at a time shall be a maximum.
(4) Find the point in the space such that the sum of whose e coordinates is 48 and whose distance from the origin is minimum.
(5) *If* $u = \frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}$, *where* $x + y + z = 1$ *prove that the stationary values of u is given by* $x = \frac{a}{a+b+c}, y = \frac{b}{a+b+c}, .z \frac{c}{a+b+c}$.

❖ **TANGENT PLANE TO THE SURFACE:**

Let the equation of the surface be **f (x, y, z) = 0**.

Then the equation of the tangent plane to the surface **f (x, y, z) = 0** at the point

P(x_1, y_1, z_1) is $(x - x_1)\left(\frac{\partial f}{\partial x}\right)_P + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_P + (z - z_1)\left(\frac{\partial f}{\partial z}\right)_P = 0.$

❖ **NORMAL LINE TO THE SURFACE:**

Let the equation of the surface be **f (x, y, z) = 0.**

Then the equation of the normal line to the surface **f (x, y, z) = 0** at the point

$P(x_1, y_1, z_1)$ is $$\frac{(x-x_1)}{\left(\frac{\partial f}{\partial x}\right)_P} = \frac{(y-y_1)}{\left(\frac{\partial f}{\partial y}\right)} = \frac{(z-z_1)}{\left(\frac{\partial f}{\partial z}\right)_P} .$$

❖ Problems Based On Above Discussion :

(1) Find the equations of the tangent plane and the normal line to the surface
 (I) $z = 3x^2 + 2y^2$ at (1, 2, 11).
 (II) $x^2 + 2y^2 + + 3z^2 = 12$ at (1, 2, -1).

(2) show that the surfaces $z = xy - 2$ *and* $x^2 + y^2 + z^2 = 3$ have the same tangent plane at (1, 1, -1).

(3) Show that the tangent plane to the surface $x^2 = y(x+z)$ at any point passes through origin.

(4) Show that the plane lx +my +nz = p touches the ellipsoid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, if\ l^2a^2 + m^2b^2 + n^2a^2 = p^2$.

❖ ERRORS AND APPROXIMATIONS :

➢ ERRORS:

(1) δx *is an error in x.*

(2) $|\delta x|$ *is called an absolute error in x.*

(3) $\frac{\delta x}{x}$ *is called the relative error in x.*

(4) $\frac{\delta x}{x} \times 100$ *is called the percentage error in x.*

➢ Let **f(x, y)** be a continuous function of x and y .

Let δx & δy be the change in **x** & **y** respectively then related change in **f(x, y)** is given by $\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$ .approximately.

❖ Problems Based On Above Discussion :

(1) Find the percentage error in the area of the rectangle when an error of One percent is measuring its length and breadth.

(2) If the kinetic energy $T = \frac{1}{2}mv^2$, find approximately the change in T as m changes from 49 to 49.5 and v changes from 1600 to 1590.

(3) If measurements of radius of base and height of a right circular cone are in error by -1 % and 2%, prove that there will be no error in volume.

(4) The acceleration of a piston is calculated by the formulae $f = w^2 r\left(\cos\theta + \frac{r}{l}\cos 2\theta\right), w, \theta$ being variables. The acceleration is calculated for $\theta = 30^0$ *and* $\frac{r}{l} = \frac{1}{4}$ .If the values of w, θ are each 1% small, show that the calculated value of f is about 1.5% too small.

(5) find the approximate value of

(I) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at point (3.01, 4.02, 11.98) .

(II) $\sqrt[4]{\left[(1.9)^3 + (2.1)^3\right]}$

(III) $\sqrt[5]{\left[(3.82)^2 + 2(2.1)^3\right]}$

Prepared by Mr.Zalak Patel
Lecturer, Mathematics