PARTIAL DIFFERENTIAL EQUATIONS

Weightage for university exam: 18Marks

- **Def ^{n} :**

  A diff. equation which involves partial derivatives is called partial differential equation. for e.g. $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$.

- **Formation of a differential equation:**

  To form a differential equation from a given relationship of variables we will eliminate arbitrary constants or arbitrary functions of these variables using differentiation.

- **Solution of a partial differential Equation:**

  A solution or integral of a partial differential equation is a relation between the variables, which is free from derivatives and satisfies the partial differential equation.

  For e.g. $u = e^x \sin y$ is a solution of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

- **Complete Integral:**

  The solution which contains a number of arbitrary constants equal to the order of the differential equation is called a complete integral.

- **Methods To Solve Partial Differential Equations:**

  - **Method of Direct Integration:**

    This method is applicable to those problems, where direct Integration is possible. This solution depends on definition of P.D.E.

  - **Lagrange's equation :**

    The partial differential equation **Pp + Qq =R,**
    Where P, Q, R are functions of x, y, z.
    is known as Lagrange's (linear) equation.

    The general solution of the partial differential equation **Pp + Qq =R,** is

    $\phi\,(u(x,y,z),\ v(x,y,z)) = 0$,

    Where ϕ is an arbitrary function and $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ are two independent solutions of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ (subsidiary auxiliary equations).

❖ Solutions of Nonlinear Partial Differential Equations:

Standard forms:

(1) **Equations involving only P and Q:**

Form : $f(p,q)=0$

Method to get solution : Replace **p** by a and **q** by b in $f(p,q)=0$
Get form $f(a,b)=0$ and find relation b = F(a).

Solution : z = ax + F (a) + c.

(2) **Equations not involving the independent variables:**

Form : $f(z,p,q)=0$ ….(1)

Method to get solution : assume q = ap.
Then the (1) becomes $f(z,p,ap)=0$
Get form p = $\phi(z,a)$
Use dz = $\phi(z,a)$ dx + a$\phi(z,a)$ dy

Solution : $\int \frac{dz}{\phi(z,a)}$ = x + ay + b.

(3) **Separable Equations:**

Form : $f_1(x,p)=f_2(y,q)$ ….(1)

Method to get solution : assume $f_1(x,p)$ = a = $f_2(y,q)$
Solving $f_1(x,p)$ = a, get **p** = $\phi_1(x,a)$
Solving $f_2(y,q)$ = a, get **p** = $\phi_2(y,a)$
Use dz = $\phi_1(x,a)$ dx + $\phi_2(y,a)$ dy

Solution : z = $\int \phi_1(x,a)$ dx + $\int \phi_2(y,a)\,dy$ + b.

(4) **Clairaut's Form:**

Form : z = px +qy + $f(p,q)$ ….(1)

Method to get solution : Replace **p** by **a** and **q** by **b** in (1)

Solution : z = ax + by + $f(a,b)$.

(5) **Equations Reducible to standard Forms:**

This category includes those p.d.e. which do not fall directly under any of the above four forms

We can reduce our given equations in one of the above four forms

- **Homogeneous Linear Partial Differential Equations With Constant Coefficient:**

A diff. equation

$$\frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \ldots\ldots\ldots\ldots\ldots\ldots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

Where $a_1, a_2, \ldots\ldots\ldots, a_n$ are constants

is called Linear partial differential equation of nth order first degree.

Above equation can be written by

$$(D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots\ldots\ldots\ldots + a_n D'^n)z = F(x, y) \quad \ldots\ldots(1)$$

Where D = $\frac{d}{dx}$ and D’ = $\frac{d}{dy}$

(1) can be written as **f (D,D’) z= F(x,y)**

- **General Solution of Differential Equation:**

General solution = Complementary function + Particular integral

G.S. = C.F. + P.I.

- **Complementary function (C.F.)of Differential Equation**

The solution which contains a number of arbitrary constants equal to the order of the differential equation is called the **complementary function (C.F.)** of a Differential equation.

- **Auxiliary Equation:**

An equation **f (D,D’) = 0** is Auxiliary Equation .

- **Method to find C.F. :**

Solve equation $(D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots\ldots\ldots\ldots + a_n D'^n)z$ = o
for values of D & D’.

Say factorization is $(D - m_1 D')(D - m_2 D')(D - m_3 D')\ldots\ldots\ldots(D - m_n D') = 0$.

Then we can classify roots and we can find C.F. by following way

Sr. No.	Classification of roots	C.F.
1	If roots D = $m_1, m_2, m_3, \ldots\ldots\ldots, m_n$ of Auxiliary equation are real and distinct	$y = f_1(y + m_1 x) + f_2(y + m_2 x) + \ldots\ldots\ldots$
2	Two roots are equal (real roots) i.e. If roots are $m_1 = m_2$, $m_3 \ldots\ldots, m_n$	$y = f_1(y + m_1 x) + x f_2(y + m_1 x)$ $+ f_3(y + m_3 x)\ldots\ldots\ldots$

- **Method to find P.I. :**

Consider f (D, D’) z = F(x, y)

Particular integral can be given by P.I. = $\frac{1}{f(D, D')} F(x, y)$

❖ Short cut methods to find P.I.:

Let given **L.P.D.E.** is**. f (D, D') z = F(x, y)**
We can use following short cut methods to find P.I.

(1) $$\frac{1}{f(D,D')}e^{ax+by}=\frac{1}{f(a,b)}e^{ax+by} \quad if\ f(a,b)\neq 0.$$

(2) $$\frac{1}{f(D^2,DD',D'^2)}\sin(ax+by)=\frac{1}{f(-a^2,-ab,-b^2)}\sin(ax+by)$$
$$if\ f(-a^2,-ab,-b^2)\neq 0$$

and

$$\frac{1}{f(D^2,DD',D'^2)}\cos(ax+by)=\frac{1}{f(-a^2,-ab,-b^2)}\cos(ax+by)$$
$$if\ f(-a^2,-ab,-b^2)\neq 0$$

(3) $$\frac{1}{f(D,D')}x^m y^n=\frac{1}{1+\phi\left(\frac{D}{D'}\right)}x^m y^n$$

Use $\frac{1}{1+t}=1-t+t^2-t^3+\ldots\ldots\ldots\ldots\ldots\ldots$

Or $\frac{1}{1-t}=1+t+t^2+t^3+\ldots\ldots\ldots\ldots\ldots\ldots$ On $\frac{1}{1+\phi(D)}$ and use

$\frac{1}{D}F(x,y)=\int F(x,y)\ dx\ \ and\ \frac{1}{D'}F(x,y)\,dy$ to get P.I.

(4) $$\frac{1}{f(D,D')}e^{ax+by}\phi(x,y)=e^{ax+by}\frac{1}{f(D+a,D'+b)}\phi(x,y)$$

(5) General method to find P.I.:

P.I. = $\frac{1}{f(D,D')}F(x,y)$

$$=\frac{1}{(D-m_1D')(D-m_2D')(D-m_3D')\ldots\ldots\ldots\ldots(D-m_nD')}F(x,y)$$

$$=\left\{\frac{1}{(D-m_1)(D-m_2)(D-m_3)\ldots\ldots\ldots\ldots}\right\}\frac{1}{D-m_1}F(x,y)$$

$$=\left\{\frac{1}{(D-m_1)(D-m_2)(D-m_3)\ldots\ldots\ldots\ldots}\right\}\int F(x,c-m_1x)\ dx$$

Where $\mathbf{c_1}$ is replaced by $\mathbf{y + m_1x}$.
Continue above process for all factors of $f(D)$. And get P.I. for y.