BETA AND GAMMA FUNCTIONS

❖ **Introduction to topic** :

Weightage for university exam: 06Marks
No. of lectures required to teach: 02hrs

❖ **Definition of Gamma function :**

The gamma function is denoted and defined by the integral

$$\Gamma(m) = \int_0^{\infty} e^{-x} x^{m-1} dx \ (m > 0)$$

❖ **Properties of Gamma function :**

(1) $\Gamma(m+1) = m\Gamma(m)$

(2) $\Gamma(m+1) = m!$ when m is a positive integer.

(3) $\Gamma(m+a) = (m+a-1)(m+a-2)\ldots\ldots a\,\Gamma(a)$,when n is a positive integer.

(4) $\Gamma(m) = 2\int_0^{\infty} e^{-x^2} x^{2m-1} dx \ (m > 0)$

(5) $\dfrac{\Gamma(m)}{t^m} = \int_0^{\infty} e^{-tx} x^{m-1} dx \ (m > 0)$

(6) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(7) $\int_0^{\infty} e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$.

❖ **Definition of Beta function :**

The beta function is denoted and defined by the integral

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \ (m > 0, n > 0)$$

$B(m,n)$ is also denoted by $\beta(m,n)$.

❖ **Properties of Beta function :**

(1) $B(m,n) = B(n,m)$

(2) $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

(3) $B(m,n) = \int_0^{\infty} \dfrac{x^{m-1}}{(1+x)^{m+n}} dx$

(4) $B(m,n) = \int_0^{1} \dfrac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx$

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❖ Relation between Beta and Gamma functions:

- Relation between Beta and Gamma functions is

$$B(m,n)=\frac{\Gamma(m)\,.\,\Gamma(n)}{\Gamma(m+n)}$$

❖ Using above relation we can derive following results

- $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\, B\left(\frac{p+1}{2},\frac{q+1}{2}\right)=\frac{\Gamma\left(\frac{p+1}{2}\right).\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

- $$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

- **Euler's formula:** $\Gamma(n)\,.\,\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}$

- **Legendre's formula (or Duplication formula):**

$$\Gamma(n)\,.\,\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}\;\Gamma(2n)}{2^{2n-1}}$$